

Transmission and reflection of phonons and rotons at the superfluid helium-solid interface.

I. N. Adamenko,^{1,*} K.E. Nemchenko,¹ and I. V. Tanatarov^{2,†}

¹Karazin Kharkov National University, 4 Svobody Sq., Kharkov 61077, Ukraine

²Akhiezer Institute for Theoretical Physics, NSC KIPT of NASU, Akademicheskaya St. 1, Kharkov, 61108, Ukraine.

We solve the problem of the transmission and reflection of phonons and rotons at the interface between superfluid helium and a solid, for all angles of incidence and in both directions. A consistent solution of the problem is presented which allows us to rigorously describe the simultaneous creation of phonons, R^- , and R^+ rotons in helium by either a phonon from the solid or a helium quasiparticle incident on the interface. The interaction of all *HeII* quasiparticles with the interface, and their transmission, reflection and conversion into each other, is described in a unified way. The angles of propagation and the probabilities of creating quasiparticles are obtained for all cases. Andreev reflection of helium phonons and rotons is predicted. The energy flows through the interface due to phonons, R^- , and R^+ rotons are derived. The small contribution of the R^- rotons is due to the small probability of an R^- roton being created by a phonon in the solid, and vice versa. This explains the failure to directly create beams of R^- rotons prior to the experiments of Tucker and Wyatt in 1999. New experiments for creating R^- rotons, by beams of high-energy phonons (h-phonons), are suggested.

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I. INTRODUCTION

Many physical properties of continuous media at low temperatures can be described in terms of quasiparticles. The quasiparticles of superfluid helium are called phonons, R^- rotons, and R^+ rotons. They have a non-monotonic dispersion curve and the R^- rotons have a negative group velocity, i.e. their momentum is directed opposite to the group velocity, see Fig.1. The phonons and rotons are observed in many experiments, such as in neutron scattering in helium [1] and in the direct experiments [2, 3] where beams of superfluid helium quasiparticles are created by a heated solid. The quasiparticles propagate in the helium, and interact and reflect from different surfaces. Also they quantum evaporate helium atoms from the free surface. These have been investigated both experimentally and theoretically (see for example [4–8]). Interestingly, R^- rotons were not detected in direct experiments until 1999, when they were finally created by a specially constructed source [3]. They were observed by quantum evaporation. All the earlier attempts to create R^- , with ordinary solid heaters, were unsuccessful.

The problem of the interaction of rotons in superfluid helium with interfaces, their reflection, transmission, and mode change, was first considered in [9]. However, the method used there did not take into account the simultaneous creation of phonons, R^+ and R^- rotons by a phonon in the solid incident on the interface, and it could not distinguish between the R^+ and R^- rotons. Later in [10] it was shown that R^- rotons cannot be created at the interface with a solid by a phonon from the solid, provided we can neglect the possibility of creation of the other quasiparticles in the same process at the interface. In the current work a consistent solution is introduced, which allows us to rigorously solve the problem of the simultaneous creation of phonons and rotons. Also it describes in a unified way, the interaction of all

HeII quasiparticles with the interface: their transmission, reflection and conversion into each other. These are the fundamental elementary processes that determine the heat exchange between *HeII* and a solid, and the associated phenomena, such as the Kapitza temperature jump (see for example [11]). We investigate all these phenomena. The probability of creation of each quasiparticle at the interface is derived for all cases. The failures of attempts to detect R^- rotons prior to experiments [3] is explained, and predictions are made for new experiments on the interaction of phonons and rotons with a solid and the creation of R^- rotons at the interface by high energy phonons (h-phonons).

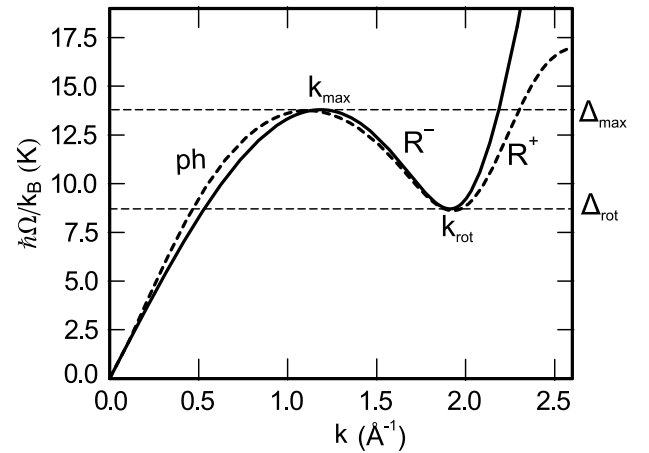


FIG. 1. The solid line is $\Omega(k)$ from (1) for $s = 230.7 \text{ m/s}$, $k_g = 1.9828 \text{ \AA}^{-1}$, and $\lambda = -0.9667$; the dashed line is the measured dispersion curve of superfluid helium [1] at the saturated vapour pressure.

We describe superfluid helium with its distinctive dispersion relation $\Omega(k)$, with the maximum and roton minimum, within the framework of the theory developed in [10]. The quantum fluid is considered as a continuous medium at all length scales. This model is based on the fact that the thermal de Broglie wavelength of a particle of a quantum fluid exceeds the average inter-

* i.n.adamenko@mail.ru

† igor.tanatarov@gmail.com

atomic separation. Then the variables of the continuous medium can only be assigned values, at each mathematical point of space, in a probabilistic sense.

The idea to describe superfluid helium as a continuous medium at microscopic scales has been successfully used for decades. Atkins [12] used it in the 1950s to describe the mobility of electrons and ions in *HeII*, when he introduced bubbles and snowballs of microscopic size. Lately the vortices in superfluid helium with cores of sizes of the scale of interatomic distances are being studied extensively, see for example Ref.[13] and the references cited there. Some recent simulations on the dynamics of atoms in helium nanodroplets [14, 15] also affirm that *HeII* is well described as a continuous medium at microscopic scales.

As shown in [10], application of the methods of theory of continuous medium at microscopic scales demands the relations between the variables of continuous medium to become nonlocal. In the work [16], the nonlocal hydrodynamics was introduced to describe small oscillations in superfluid helium, and in [17, 18] it was used to describe ripplon-roton hybridization and dispersion relation of ripplons. The nonlocality allows one to analyse a continuous medium with an arbitrary dispersion relation. This possibility was discussed in [19]. However, the theoretical justification of this approach remained on the intuitive level until the work [10].

In this paper, following [10], the quasiparticles are described as wave packets propagating in the superfluid. The long wavelength excitations are phonons, while the short wavelength ones are rotons. R^- rotons correspond to the descending part of the dispersion curve and have negative group velocity, i.e. they propagate in the direction opposite to their momentum. This simple model allows us to use the methods of the theory of continuous medium and avoid the difficulties that appear in other phenomenological models, such as [17, 18].

So, with the help of boundary conditions for the continuous media at the interface, we find the creation probabilities of all the quasiparticles' creation when any of them is incident on the interface, as the energy reflection and transmission coefficients for the corresponding wave packets. The method allows us to obtain the analytical expressions for the probabilities as functions of angles and frequency.

In section 2 of the paper we formulate the problem and obtain the general solution of the nonlocal equations of the quantum fluid in the half-space. We consider a parameterised dispersion relation that is a good approximation to the measured dispersion curve of superfluid helium. The solution is sought in the form that generalizes the solution for a monotonic dispersion of a general form obtained in [20]. The consequences of using the boundary conditions are discussed, these include multiple critical angles, backward refraction and retro-reflection (or Andreev reflection [21]) of phonons and rotons (see also [18]).

In section 3 the boundary conditions are used to derive both the amplitude and energy reflection and transmission coefficients for any incident wave for arbitrary incidence angles. The preliminary results for rotons at normal incidence were discussed in the authors' report at the conference [22].

In section 4 the energy flows through the interface due to phonons, R^- , and R^+ rotons are calculated as functions of temperature. The contribution of the R^- rotons

to the energy flows, in both directions, are shown to be small. This means that R^- rotons are hardly created by a solid heater and are poorly detected by a solid bolometer. This explains why R^- rotons could not be detected in direct experiments until the work [3]. There they were created by a source made up of two heaters facing each other which allowed mode changes, and detection was achieved by quantum evaporation.

The results obtained in this work can be used in other fields of physics. In particular they are important for classical acoustics, where the problem of wave transmission through an interface has been solved only for the case when the dispersion relations of both adjacent media are strictly linear (see, for example, [23]), let alone non-monotonic. This problem concerning real media, with nonlinear dispersion, was of interest in the middle of the last century [24] and is still relevant today [25, 26].

II. DERIVATION OF EQUATIONS AND THEIR SOLUTION

A. Problem Formulation

Let us consider two continuous media separated by a sharp interface $z = 0$. In the region $z < 0$ there is an ordinary continuous medium with sound velocity s_{sol} and equilibrium density ρ_{sol} . For the solid we only take into account longitudinal waves.

Taking into account the transverse waves can be done in the same framework of theory of continuous medium and does not present any difficulty. However, the calculations become much more cumbersome, while on the whole the situation does not change. Due to the very small impedance of the solid-helium interface (see section 2.4 and below), the reflection coefficients hardly change at all. For the transmitted waves additional critical angles appear, corresponding to the sound velocity of the transverse waves. Also it should be noted that taking into account both the longitudinal and transverse waves in the solid allows one to consider the contribution of Rayleigh waves, which give contributions to the transmission coefficients of He II quasiparticles into the solid at fixed incidence angles. For phonons with linear dispersion this problem was solved in [27]. This problem for the helium-solid interface may be the subject of next paper.

The region $z > 0$ is filled with the quantum fluid with equilibrium density ρ_0 and dispersion relation $\Omega(k)$ such that

$$\Omega^2(k) = s^2 k^2 \left\{ 1 + 2\lambda \frac{k^2}{k_g^2} + \frac{k^4}{k_g^4} \right\}. \quad (1)$$

Here s is sound velocity at zero frequency, k_g is wave vector that determines the scale of the curve and λ determines the form of the curve. For a range of parameters, this relation is a good approximation to the measured non-monotonic dispersion relation of superfluid helium (see Fig.1). For $\lambda < -1$ there are real k such that $\Omega^2(k) < 0$, which is nonphysical, and for $\lambda > -\sqrt{3}/2$ the curve is monotonic. For $\lambda \in (-1, -\sqrt{3}/2)$ the curve has the roton minimum at $k = k_{rot}$ and maxon maximum at $k = k_{max}$ as it should. We adopt the following set of values: $s = 230.7$ m/s, $k_g = 1.9828 \text{ \AA}^{-1}$, and

$\lambda = -0.9667$. Then the dispersion curve has the following parameters: the coordinates of roton minimum $k_{rot} = 0.9670k_g = 1.913 \text{ \AA}^{-1}$ and $\Delta_{rot} = \hbar\Omega(k_{rot})/k_B = 8.712 \text{ K}$ (k_B is the Boltzmann constant); the maxon maximum is $\Delta_{max} = \hbar\Omega(k_{max})/k_B = 13.8 \text{ K}$. These values are the experimentally measured parameters of superfluid helium dispersion curve at the saturated vapor pressure [1].

We describe this quantum fluid by nonlocal hydrodynamics as developed in Ref. [10]. Accordingly, the quantum fluid, as well as the ordinary fluid on the other side of the interface, obeys the linearized equations of continuous media

$$\frac{\partial \rho}{\partial t} = -\rho_0 \nabla \mathbf{v} \quad ; \quad \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_0} \nabla P, \quad (2)$$

where \mathbf{v} is hydrodynamic velocity, ρ and P are the deviations of density and pressure from the respective equilibrium values (for brevity we refer to them below as just density and pressure). The difference is that pressure and density in the quantum fluid are related through the non-local relation

$$\rho(\mathbf{r}) = \int_{z' > 0} d^3 r' h(|\mathbf{r} - \mathbf{r}'|) P(\mathbf{r}'), \quad (3)$$

in which the integration domain is the region filled by the quantum fluid [28].

The suggested model well describes the interface between superfluid helium and a solid, because for solids the relationships are local. In the frequency range of the dispersion curve of HeII, the dispersion laws of most solids, such as the heater materials of copper or gold, are very close to linear and they can be described as ordinary continuous media.

Equations (3) and (2) lead to the integro-differential equation for pressure

$$\Delta P(\mathbf{r}, t) = \int_{z' > 0} d^3 r' h(|\mathbf{r} - \mathbf{r}'|) \ddot{P}(\mathbf{r}', t), \quad (4)$$

that is set for $x, y, t \in (-\infty, \infty)$, $z \in (0, +\infty)$. In the infinite medium, when the integration and definition domains are infinite, the Fourier transform of (4) gives us the relation between the Fourier transform of the kernel $h(r)$ and the dispersion relation of the fluid $\Omega(k)$ [10]:

$$h(k) = \frac{k^2}{\Omega^2(k)}. \quad (5)$$

For the dispersion relation (1) we obtain from the Fourier transform of Eq. (5):

$$h(r) = \frac{k_g^4}{4\pi s^2 r} \frac{1}{k_+^2 - k_-^2} (e^{ik_+ r} - e^{-ik_- r}), \quad (6)$$

where k_+ and $(-k_-)$ are the poles of $h(k)$ in the upper half-plane \mathbf{C}_+

$$k_{\pm} = k_g \left(\sqrt{1 - \lambda} \pm i\sqrt{1 + \lambda} \right) / \sqrt{2}, \quad k_+ = k_-^* \in \mathbf{C}_+. \quad (7)$$

Here the asterisk denotes the complex conjugate and \mathbf{C}_+ is the upper complex half-plane. Due to the last condition, the kernel, Eq. (6), despite the complex notation, is real.

There is no convolution product in Eq. (4), either in the sense of one- or two-sided Fourier transform or Laplace transform, because the lower limit by z' is finite while the kernel is symmetrical $h(\mathbf{r}) = h(r)$.

We consider the problem of waves transferring through the interface. As the equations (2) are local and coincide with the notation used in the equations of ordinary ideal continuous medium, the two boundary conditions on the interface (local!) are obtained from their integral forms, in the usual way, using the theory of continuous medium:

$$\begin{cases} P(x, y, z = -0, t) = P(x, y, z = +0, t), \\ V_z(x, y, z = -0, t) = V_z(x, y, z = +0, t). \end{cases} \quad (8)$$

By applying the solutions of the equations of continuous media, on the both sides of the interface, the boundary conditions (8) give us the solution in the whole space and thus provide us with all the coefficients of reflection and transmission. The solution in the solid is well-known, and the solution in the quantum fluid is derived in the following subsection.

B. Solution of Eq. (4) in half-space

The equation that determines the relation between k and ω

$$\Omega^2(k) = \omega^2 \quad (9)$$

with $\Omega^2(k)$ from Eq. (1) is sixth order with respect to k . Its six roots are functions of ω and are denoted k_μ for $\mu = 1, \dots, 6$. We note that if we used more terms in the polynomial $\Omega^2(k)$, then the higher order equation for k would give 6 real roots and the other roots would be imaginary.

In the problem of waves transferring through the interface, the two boundary conditions (8) can be satisfied for all x, y , and t only if all of the waves present on both sides of the interface have the same frequency ω and transverse component of wave vector \mathbf{k}_τ . A single monochromatic wave is not a solution of Eq. (4). Therefore, as there are in total six roots of Eq. (9), we search for the solution as a sum of six monochromatic waves, with the same frequency ω and transverse component of wave vector k_τ (the y axis is chosen along \mathbf{k}_τ), i.e. with the form

$$P(\mathbf{r}, t) = \sum_{\mu=1}^6 A_\mu \exp[i(\mathbf{k}_\mu \mathbf{r} - \omega t)]. \quad (10)$$

Here the vectors \mathbf{k}_μ are

$$\mathbf{k}_\mu = k_{\mu z} \mathbf{e}_z + k_\tau \mathbf{e}_y, \quad (11)$$

$$k_\mu^2 = k_{\mu z}^2 + k_\tau^2. \quad (12)$$

The transverse component k_τ is real for physical reasons, but k_μ and $k_{\mu z}$ can be either real or complex, as we have to ensure the boundedness of our solution only in the half-space $z > 0$.

After substitution of Eq. (10) into Eq. (4), we obtain the system of equations for the amplitudes A_μ and the equations for $k_{\mu z}$, as functions of ω and k_τ . The system

for A_μ is

$$\begin{cases} \sum_{\mu=1}^6 A_\mu (k_{\mu z} - k_{+z})^{-1} = 0, \\ \sum_{\mu=1}^6 A_\mu (k_{\mu z} + k_{-z})^{-1} = 0, \end{cases} \quad (13)$$

where

$$k_{\pm z}^2 = k_\pm^2 - k_\tau^2, \quad k_{+z} = k_{-z}^* \in \mathbf{C}_+. \quad (14)$$

The equations for $k_{\mu z}$ are reduced to the form

$$\Omega^2 (k_\mu^2 = k_{\mu z}^2 + k_\tau^2) = \omega^2 \quad \text{for } \mu=1, \dots, 6. \quad (15)$$

The system of two homogeneous equations for the amplitudes A_μ (13) ensure that no (nontrivial) solutions exist with less than 3 non-zero amplitudes A_μ , i.e. there are no eigensolutions of Eq. (4), in the half-space, consisting of less than 3 monochromatic waves. This is the consequence of the nonlocality, which changes equation (4) itself in the presence of the interface. In infinite space, on the contrary, the domain of integration is the whole space, and the equation is solved by the Fourier transform, and its solution is a superposition of plane waves with dispersion (1).

A solution of (4), with the smallest possible number of waves being three, is constructed in the form (10), with three terms out of six, by picking a subset of any three different roots $\{k_\alpha, k_\beta, k_\gamma\}$ out of the set of six $\{k_\mu\}_{\mu=1, \dots, 6}$. Then it can be rewritten with the help of Eqs. (13) in the form that contains a single amplitude:

$$\begin{aligned} P_{\{k_\alpha, k_\beta, k_\gamma\}}(\mathbf{r}, t) &= P_{\alpha\beta\gamma}^{(0)} \times \\ &\times \left\{ \frac{(k_{\alpha z} - k_{+z})(k_{\alpha z} + k_{-z})}{(k_{\alpha z} - k_{\beta z})(k_{\alpha z} - k_{\gamma z})} e^{ik_{\alpha z} z} + \right. \\ &+ \frac{(k_{\beta z} - k_{+z})(k_{\beta z} + k_{-z})}{(k_{\beta z} - k_{\gamma z})(k_{\beta z} - k_{\alpha z})} e^{ik_{\beta z} z} + \\ &\left. + \frac{(k_{\gamma z} - k_{+z})(k_{\gamma z} + k_{-z})}{(k_{\gamma z} - k_{\alpha z})(k_{\gamma z} - k_{\beta z})} e^{ik_{\gamma z} z} \right\} e^{i(k_\tau y - \omega t)}, \end{aligned} \quad (16)$$

where $P_{\alpha\beta\gamma}^{(0)}$ is chosen so that $P_{\{k_\alpha, k_\beta, k_\gamma\}}(\mathbf{r} = 0, t = 0) = P_{\alpha\beta\gamma}^{(0)}$.

The velocity is obtained from Eqs. (16) and (2):

$$\begin{aligned} \mathbf{v}_{\{k_\alpha, k_\beta, k_\gamma\}}(\mathbf{r}, t) &= \frac{P_{\alpha\beta\gamma}^{(0)}}{\rho_0} \times \\ &\times \left\{ \frac{\mathbf{k}_\alpha}{\omega} \frac{(k_{\alpha z} - k_{+z})(k_{\alpha z} + k_{-z})}{(k_{\alpha z} - k_{\beta z})(k_{\alpha z} - k_{\gamma z})} e^{ik_{\alpha z} z} + \right. \\ &+ \frac{\mathbf{k}_\beta}{\omega} \frac{(k_{\beta z} - k_{+z})(k_{\beta z} + k_{-z})}{(k_{\beta z} - k_{\gamma z})(k_{\beta z} - k_{\alpha z})} e^{ik_{\beta z} z} + \\ &\left. + \frac{\mathbf{k}_\gamma}{\omega} \frac{(k_{\gamma z} - k_{+z})(k_{\gamma z} + k_{-z})}{(k_{\gamma z} - k_{\alpha z})(k_{\gamma z} - k_{\beta z})} e^{ik_{\gamma z} z} \right\} e^{i(k_\tau y - \omega t)}. \end{aligned} \quad (17)$$

As the two conditions (13) restrict the number of free amplitudes in Eq. (10) from six to four, any four linear-independent solutions of the form (16) constitute the basis set of solutions of Eq. (4) for given ω and k_τ , and any solution consisting of 4, 5, or 6 monochromatic waves can be represented as their linear combination.

C. Roots of dispersion equation. In- and out-solutions

The roots of Eq. (9) with $\Omega^2(k)$ from Eq. (1) with respect to k^2 are $k_i^2 = k_g^2 \xi_i$ for $i=1, 2, 3$, where ξ_i are the three dimensionless roots of the cubic equation

$$\xi^3 + 2\lambda\xi^2 + \xi - \chi^2 = 0. \quad (18)$$

Here $\chi = \omega/(sk_g)$ is the dimensionless frequency; $\xi_i(\lambda, \chi)$ are some elaborate complex-valued functions.

The most interesting frequency range is $\chi \in (\chi_{rot}, \chi_{max})$, where $\chi_{rot}(\lambda)$ and $\chi_{max}(\lambda)$ are the dimensionless frequencies that correspond to the roton minimum and maxon maximum respectively. For such frequencies there are three types of running waves in the quantum fluid, corresponding to phonons, R^- , and R^+ rotons. The branches are numbered in this case in the ascending order of the absolute values of their wave vectors k_i : $0 < k_1 < k_{max} < k_2 < k_{rot} < k_3$, so that $i=1$ corresponds to phonons, $i=2$ to R^- rotons, and $i=3$ to R^+ rotons.

We now consider the problem of quasiparticles transfer through the interface. The quasiparticles are treated as wave packets that propagate in the two media. Therefore, when we build the solutions in the quantum fluid, we have to take into account that wave packets, as well as quasiparticles, propagate with their group velocities $d\Omega/d\mathbf{k}$ [29]. So, a wave packet of the quantum fluid composed of waves with wave vectors close to \mathbf{k}_0 , with its length $k_0 < k_{max}$ (so that it is a phonon wave packet) and the z th component $k_{0z} > 0$, propagates away from the interface; but a wave packet, composed of waves with wave vectors close to one with length $k_0 \in (k_{max}, k_{rot})$ (so it is an R^- roton packet) and the z th component $k_{0z} > 0$, propagates towards the interface.

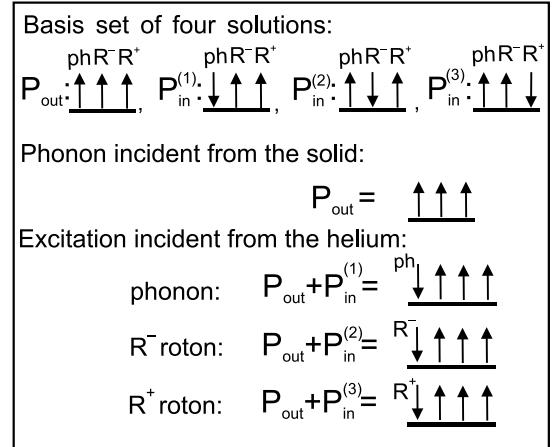


FIG. 2. The basis set of solutions in the superfluid helium in the half-space and their sums, which correspond to the different incident excitations.

Let us construct the solution in the quantum fluid P_{out} (the “out-solution”) that is realized when a wave in the solid is incident on the interface. This solution should contain only such waves that constitute wave packets traveling away from the interface (i.e. waves with positive group velocity) or waves that are damped at $z \rightarrow +\infty$. Picking 3 out of 6 vectors \mathbf{k}_μ is the same as picking their normal components $k_{\mu z}$, as they all have the same k_τ . The six normal components $k_{\mu z}$, obtained as solutions of

Eq. (15), are grouped into three pairs of roots $\pm\sqrt{k_i^2 - k_\tau^2}$ for $i = 1, 2, 3$. For each pair with the same i , either both roots are real which occurs for small enough angles $k_\tau < k_i$, or both roots are imaginary for $k_\tau > k_i$. In the first case one root corresponds to a wave traveling towards the interface, the other to a wave traveling away from it. The second case, one root gives a damped wave in $z > 0$, while the other gives an exponentially unbounded wave. So, P_{out} contains no more than three waves (and no less because there are no such solutions) and therefore has the form of Eq. (16) (see Fig. 2). The squared normal components of the three constituent waves are

$$k_{iz}^2 = k_i^2 - k_\tau^2 \quad \text{for } i = 1, 2, 3. \quad (19)$$

We define the signs of roots k_{iz} for P_{out} to be made up of waves with the normal components of wave vectors equal to k_{1z} , k_{2z} and k_{3z} . Then taking into account the negative group velocity of R^- rotons [29], for the signs of real k_{iz} we obtain $k_{1z}, k_{3z} > 0$ and $k_{2z} < 0$. If $k_\tau > k_i$ for some i , the corresponding wave $\sim \exp(ik_{iz}z)$ is bounded in $z > 0$ for $\text{Im } k_{iz} > 0$. Then in the general case we have

$$\begin{aligned} k_\tau \in (0, k_1) : 0 < k_{1z} < (-k_{2z}) < k_{3z} \\ k_\tau \in (k_1, k_2) : 0 < (-k_{2z}) < k_{3z}, \quad k_{1z} \in \mathbf{C}_+ \\ k_\tau \in (k_2, k_3) : 0 < k_{3z}, \quad k_{1z}, k_{2z} \in \mathbf{C}_+. \end{aligned} \quad (20)$$

Then the out-solution has the form of Eq. (16) with the three wave vectors picked from the set of six with normal components k_{1z} , k_{2z} and k_{3z} :

$$P_{out} = P_{\{k_\alpha, k_\beta, k_\gamma\}} [k_{\alpha z} = k_{1z}, k_{\beta z} = k_{2z}, k_{\gamma z} = k_{3z}]. \quad (21)$$

In order to solve the problem of a wave transferring through the interface from superfluid helium into the solid, we need also solutions containing waves that are traveling towards the interface (i.e. wave packets comprised of these waves should be traveling towards the interface). We define them in the way, which is illustrated by Fig. 2. The solution $P_{in}^{(1)}$ is constructed of waves with z th components of wave vectors $(-k_{1z})$, k_{2z} and k_{3z} , with the amplitudes related through Eqs. (13). Solution $P_{in}^{(2)}$ is constructed of waves with k_{1z} , $(-k_{2z})$ and k_{3z} . The last one, $P_{in}^{(3)}$ contains waves with k_{1z} , k_{2z} and $(-k_{3z})$. Then the three sorts of in-solutions, that correspond to the three types of the incident waves, can be written in the form

$$P_{in}^{(i)} = P_{out}|_{k_{iz} \rightarrow (-k_{iz})} \quad \text{for } i = 1, 2, 3. \quad (22)$$

The $P_{in}^{(i)}$ solution corresponds to the incident wave of type i . So, a linear combination of P_{out} and $P_{in}^{(2)}$ consists of one R^- roton wave ($i=2$) that corresponds to the R^- roton wave packet incident on the interface, and all three waves that correspond to the reflected phonon, R^- , and R^+ roton wave packets. The amplitudes of the phonon and R^+ roton waves are the sums of the amplitudes of those waves present in both P_{out} and $P_{in}^{(2)}$. This is the solution in $z > 0$ realized when an R^- roton is incident on the interface. The four solutions (21) and (22) are linearly-independent due to their structure and can be used as the basis set of solutions as mentioned at the end of the previous subsection.

For $\chi \in (\chi_{rot}, \chi_{max})$ the roots k_{iz} are fully defined by

Eqs. (19) and (20). For $\chi \in (0, \chi_{rot})$ the roots $\xi_{2,3}$ are complex and $\xi_2 = \xi_3^*$; then k_{2z} and k_{3z} are defined so that the roton waves $\sim \exp(ik_{2z,3z}z)$ are damped, so $k_{2z} = -k_{3z}^* \in \mathbf{C}_+$. In the limit $\chi \rightarrow 0$ the phonon waves have almost linear dispersion $k_1 \approx \omega/s$, and it can be shown that $k_{2z} \rightarrow k_{+z}$ and $k_{3z} \rightarrow -k_{-z}$. Therefore the amplitudes of all roton waves, that contain multipliers $(k_{2z} - k_{+z})$ and $(k_{3z} + k_{-z})$, go to zero, see (16), and the general solution tends to the ordinary superposition of incident and reflected phonon waves, with wavelengths much greater than the scale of nonlocality $|k_\pm|^{-1}$. This limiting case can also be obtained by passing to the long-wave limit $h(r) \rightarrow \delta(r)/s^2$ in Eq. (4), thus making it a local wave equation.

The equation (4) was solved earlier in work [20] for the case of arbitrary but monotonic dispersion relation. The Wiener and Hopf method was used there, the application to this type of problem was suggested in [28] and developed in [30]. It is much more general and seems to be more rigorous than the method used here, though less straight-forward. The solution of [20] can be generalized to the nonmonotonic case and can be shown to yield for the dispersion relation (1) exactly the solutions (21) and (22).

In the work [17] the problem analogous to (4) was considered in order to investigate the hybridization of rotons and ripplons. There the problem in half-space was replaced by the one in the infinite medium with symmetrical extrapolation of the solutions with respect to variable z . However, in contrast to the usual differential (local) equations, for an integral equation such as (4) the solution cannot be thus extrapolated symmetrically to $z|0$. Indeed, if we consider formally the solution of Eq. (4) in $z < 0$, it would be defined unambiguously by the solution in $z > 0$ (from Eq. (16)), through the integral of Eq. (4). Direct substitution shows that the full solution is not even. This might be the reason that the method [17] gave wrong results near the surface waves threshold and was then rejected; in the next paper by the authors [18] another approach was used for that problem.

D. Multiple critical angles and Andreev reflection

Even before applying the boundary conditions (8) and finding the solution in the whole space, we can use the fact that two linear boundary conditions for the variables of continuous media are satisfied on the interface and derive a number of important consequences. First of all, we see that the solution is always constructed in a such a way that there are in total four outgoing (i.e. reflected and transmitted) waves; one in the solid and three in the quantum fluid. This is because the four conditions on the waves amplitudes, two from Eq. (8) and two from Eq. (13), can be all satisfied only when there are at least four outgoing waves. On the other hand, they can be at most four because, either in the formulation of the problem there is only one incident wave, or the requirement that the solution is bounded when some of k_{iz} are complex.

Furthermore, the two boundary conditions (8) imply that all of the waves constituting the full solution have the same frequency ω and tangential component of wave vector k_τ . When wave i is propagating at angle θ_i to the normal to the interface, $k_\tau = k_i \sin \theta_i$. Then if one of the waves is incident, the corresponding angle and k_τ are set and all the other transmission and reflection angles are

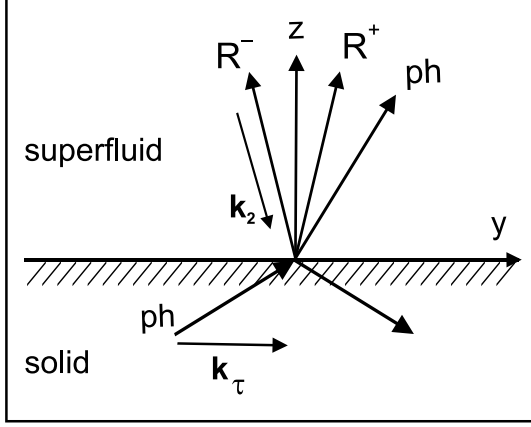


FIG. 3. When a phonon in the solid is incident on the interface, three quasiparticles, a phonon, R^- roton, and R^+ roton, are created in superfluid helium with the same ω and k_τ . The created R^- roton propagates backward in the transverse direction (i.e. retro-refracted).

determined by the generalization of Snell's law

$$\frac{\sin \theta_{sol}}{s_{sol}} = \frac{\sin \theta_1}{s_1} = \frac{\sin \theta_2}{s_2} = \frac{\sin \theta_3}{s_3}. \quad (23)$$

Here $s_i = \omega/k_i(\omega)$ are the phase velocities of the corresponding waves, that depend on frequency; $\mathbf{k}_{sol} = k_{sol} z \mathbf{e}_z + k_\tau \mathbf{e}_y$ is the wave vector of the wave in the solid and $s_{sol} = \omega/k_{sol}(\omega) = \text{const.}$ The reflection angle for the wave of the same type as the incident one is equal to the incidence angle.

From now on we will consider $s_{sol} > s$, as is the case when superfluid helium is adjacent to a solid. Usually even the strong inequality holds. Then we have

$$s_{sol} > s_1 > s_2 > s_3 > 0. \quad (24)$$

If a wave from the solid is incident at θ_{sol} , it is reflected at the same angle and the three waves are transferred into helium at angles $\theta_i < \theta_{sol}$. The R^- roton wave, as opposed to the others, due to its negative group velocity, propagates backward in the tangential direction (i.e. in the direction $y \rightarrow -\infty$, see Fig.3).

Assume a wave i is incident from helium at θ_i . Then according to Eq. (23) the transmitted wave in the solid has $\sin \theta_{sol} < 1$ for $\sin \theta_i < s_i/s_{sol}$, and if the incidence angle is greater than the critical value, $k_{sol} z$ is imaginary and the wave in the solid is exponentially damped. Thus we obtain the three angles of full internal reflection

$$\sin \theta_i^{cr} = s_i/s_{sol} \quad \text{for } i=1, 2, 3. \quad (25)$$

In the same way there are three new critical angles defined for $i > j$ (so that $s_i < s_j$):

$$\sin \theta_{ij}^{cr} = s_i/s_j < 1 \quad \text{for } \{i, j\} = \{2, 1\}, \{3, 2\}, \{3, 1\}. \quad (26)$$

If a wave i is incident and $\theta_i < \theta_{ij}^{cr}$, then wave j (with $j < i$) has $\theta_j \in (\theta_i, \pi/2)$ and $k_{jz} \in \mathbf{R}$. For $\theta_i > \theta_{ij}^{cr}$ $k_{jz}^2 < 0$, the j -th wave is damped and the corresponding quasiparticle is not created.

In an ordinary fluid, the group velocity of a wave packet is the same as the sound velocity and is constant. When a wave is incident on the interface the reflected wave has

the same wave number and due to preservation of the transverse component of the wave vector \mathbf{k}_τ , it is reflected forward. This qualitative picture is maintained in the majority of all known physical systems. However, when the reflected wave (or quasiparticle) is qualitatively different from the incident, another possibility can be realized. So, when an electron of a normal metal is incident on the interface with a superconductor, it can be retro-reflected and converted into the hole of negative effective mass that travels back along the same line as the incident electron. This effect was discovered by Andreev (see ref. [21]) and is called Andreev reflection or retro-reflection.

In our case when a helium quasiparticle is incident on the interface, three different quasiparticles are created in helium, with corresponding probabilities. If for example an R^- roton is created on the interface, along with other quasiparticles, when a phonon is incident, for the sake of brevity we will refer to this as “the phonon is reflected into R^- roton”. While phonons and R^+ rotons behave in these processes like ordinary quasiparticles, R^- rotons propagate in the direction opposite to their wave vector, due to the negative group velocity as already mentioned above. Therefore when a phonon or R^+ roton is incident on the interface, the phonons and R^+ rotons are reflected forward, while R^- rotons are reflected backward, or *retro-reflected*. In this way the transverse components of wave vectors \mathbf{k}_τ are all equal. Likewise, when an R^- roton is incident, the phonon and R^+ roton are retro-reflected. Thus we have described the effect of Andreev reflection of helium phonons and rotons.

If a monochromatic beam of phonons and rotons is incident on the interface at some angle Θ , there will be up to seven reflected beams (see Fig.4). We denote them as ij , which means “the beam of quasiparticles of type j created on the interface by the incident quasiparticles of type i ”. The beams 11, 22 and 33 are reflected forward at the incidence angle θ , and therefore constitute a single beam ii . Beams 13 and 31 are also reflected forward, while 12, 21, 32 and 23 are reflected *backward*. Due to the relations (23), the beams 32 and 21 are reflected at angles greater than θ , and beams 12 and 23 at angles less than θ .

The reflection angles depend on s_i , which are functions

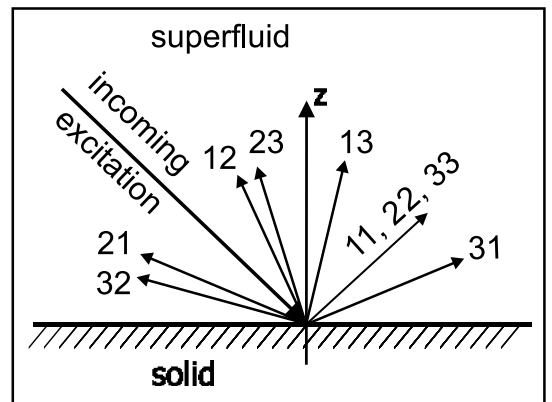


FIG. 4. When a single beam of quasiparticles is incident on the interface, a set of reflected beams is created. The beam marked ij consists of quasiparticles of type j created by incident quasiparticles of type i . The beams ii are reflected specularly, while all others propagate in different directions. The beams 12, 21, 23, and 32 are retro-reflected.

of frequency. Therefore with an incident beam which is non-monochromatic, the reflected beams all become angularly diffused [20], except for the beam *ii*. However, the relations (24) hold at all frequencies, and therefore the qualitative picture is not modified. If we place a detector on the same side from the normal as the source at greater angles, it should register the R^- rotons of beam 32 and phonons of beam 21, which were *retro-reflected*. Likewise the detector at smaller angles should register the R^- rotons of beam 12 and R^+ rotons of beam 23. Such an experiment could be carried out in order to verify qualitatively the current theory.

A successful experiment would very much depend on the intensities of the beams to be detected, and therefore on the different creation probabilities for the quasiparticles at the interface. The derivation of these probabilities is the subject of the next section.

III. REFLECTION AND TRANSMISSION COEFFICIENTS

A. Phonon in the solid incident on the interface

When a phonon in the solid is incident on the interface, it is reflected and three quasiparticles of different types are created in helium which travel away from the interface (see Fig.3). The probability of quasiparticle creation is the fraction of incident energy which is reflected or transmitted as the corresponding wave packet. Amplitude reflection and transmission coefficients can be derived in the approximation of plane waves [30].

In this approximation we consider a plane wave with frequency ω and wave number $k_{sol}(\omega) = \omega/s_{sol}$ incident on the interface at angle θ_{sol} to the normal. Then the solution in the solid is the sum of the incident and reflected waves, the solution in the quantum fluid is P_{out} from Eq. (21), which consists of three waves. All the waves have the same frequency ω and transverse component of wave vector $k_\tau = k_{sol} \cos \theta_{sol}$. The pressure amplitude of each wave P_i is the full coefficient multiplying the exponent $\exp(ik_{iz}z)$ in the out-solution. With the help of the boundary conditions (8), the amplitudes of all the waves are expressed through the amplitude of the incident wave. Then after some transformations, the amplitude reflection coefficient, defined as the ratio of pressure amplitudes in the reflected and incident waves, can be expressed in the form

$$r_{\rightarrow} = \frac{f_z - Z - i\tilde{\delta}}{f_z + Z - i\tilde{\delta}}. \quad (27)$$

Here the following notations are used. Z is a real generalization of impedance

$$Z = Z_g \cos \theta_{sol}, \quad Z_g = Z_0 \chi, \quad (28)$$

where $Z_0 = (\rho_0 s)/(\rho_{sol} s_{sol})$ is the ordinary impedance of the interface at zero frequency; $\chi = \omega/sk_g$ is the dimensionless frequency as introduced in Eq. (18); $\tilde{\delta}$ is a dimensionless constant

$$\tilde{\delta}(\lambda) = \frac{k_{+z} - k_{-z}}{ik_g} \in \mathbf{R}, \quad (29)$$

which is real due to Eq. (14);

$$f_z = f_{3z}/(k_g f_{2z}), \quad \text{where} \\ f_{nz} = k_{1z}^n (k_{2z} - k_{3z}) + k_{2z}^n (k_{3z} - k_{1z}) + k_{3z}^n (k_{1z} - k_{2z}) \\ \text{for } n=2, 3. \quad (30)$$

As $s_{sol} > s_i$, the transmission angles for all the waves θ_i are less than θ_{sol} , so $k_{iz} \in \mathbf{R}$ and f_z is a dimensionless real function of χ and k_τ/k_g .

The full transmission coefficient is $t_{\rightarrow} = 1 + r_{\rightarrow}$; the partial transmission coefficients t_i^{\rightarrow} are defined as the ratios of pressure amplitudes of each of the three waves in helium P_i to the pressure amplitude of the incident wave. They are obtained in the same way as r_{\rightarrow} :

$$t_i^{\rightarrow} = t_{\rightarrow} \frac{\psi_i}{(k_{iz} - k_{jz})(k_{iz} - k_{kz})}, \quad (31)$$

$$\text{where } \psi_i = (k_{iz} - k_{+z})(k_{iz} + k_{-z})$$

and the subscripts take values $\{i, j, k\} = \{1, 2, 3\} + \text{perm.}$ (*perm.* is for permutations). Henceforth the subscripts $\{i, j, k\}$ in the expressions of the kind of Eq. (31) take the same set of values, unless stated otherwise.

All the amplitude coefficients are complex-valued functions of frequency and incidence angle. Therefore there are always nontrivial phase shifts between the incident, reflected, and transmitted waves.

The energy reflection and transmission coefficients are the normal components of energy density flux, expressed as fractions of the incident energy flux, that are reflected or transmitted into helium. The energy density flux, in a wave packet in the quantum fluid, as shown in [29], equals the average energy density multiplied by the group velocity. It was shown in [30], that the average energy density in a wave packet or plane wave in the quantum fluid with velocity amplitude V_i is given by the same relation as in the ordinary liquid, $\rho_0 |V_i|^2$, and from Eq. (17) $V_i = P_i/(\rho_0 s_i)$. The group velocity of wave i can be obtained from Eq. (1):

$$|u_i| = \frac{s^2}{k_g^4} \frac{k_i}{\omega} |(k_i^2 - k_j^2)(k_i^2 - k_k^2)|. \quad (32)$$

Taking all of this into account and with the help of Eqs. (31), after some transformations we obtain the fractions of the normal component of the incident wave packet's energy flux that are carried by waves of each type $i=1, 2, 3$ in helium. Those are the partial energy transmission coefficients

$$D_i^{\rightarrow} = \frac{4Z}{(Z + f_z)^2 + \tilde{\delta}^2} \cdot \frac{k_{iz}}{k_g} \cdot \frac{(k_{iz} + k_{jz})(k_{iz} + k_{kz})}{(k_{iz} - k_{jz})(k_{iz} - k_{kz})}. \quad (33)$$

We note that $D_i^{\rightarrow} > 0$ for all $i=1, 2, 3$. The full energy transmission coefficient can be expressed in the form

$$D_{\rightarrow} = \sum_{i=1}^3 D_i^{\rightarrow} = \frac{4Z f_z}{(Z + f_z)^2 + \tilde{\delta}^2}. \quad (34)$$

The energy reflection coefficient is $R_{\rightarrow} = |r_{\rightarrow}|^2$. Then from Eqs. (27) and (34) after some algebraic transformations we can show explicitly that

$$R_{\rightarrow} + D_{\rightarrow} = 1, \quad (35)$$

and so energy is conserved when waves go through the

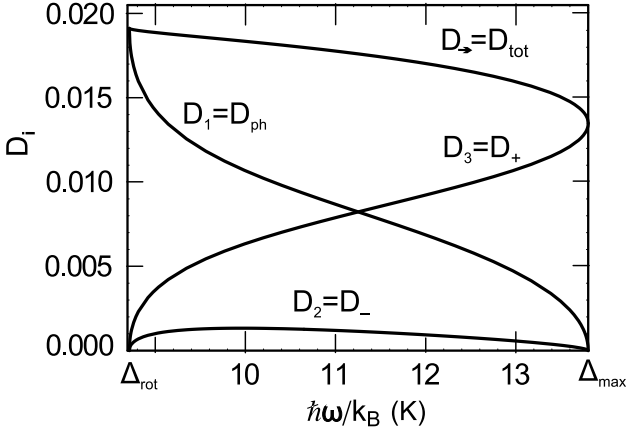


FIG. 5. Energy dependence of the transmission coefficients at $\theta=0$, with energy in temperature units, for the parameterised dispersion curve (see Fig.1) and $Z_0=0.01$. The R^- roton creation probability, D_2 , is small.

interface. This also verifies that the coefficients D_i^+ and R_{\rightarrow} are the probabilities of the creation of the corresponding quasiparticles at the interface.

Superfluid helium has a very small density and sound velocity, such that at the interface with a solid, the strong inequalities $s \ll s_{sol}$ and $\rho_0 \ll \rho_{sol}$ hold. Then, taking into account Eq. (24), we have a set of small parameters

$$Z_0 \ll 1, \quad s_i/s_{sol} \ll 1. \quad (36)$$

It can be shown that, due to the first condition of Eq. (36), in the sums of Eqs. (27), (34) and (33) the quantity Z can be neglected in comparison with the other terms. The second condition of Eq. (36) implies that, due to Eqs. (23), all the transmission angles into the helium are very small, as is indeed well-known for the interfaces between $HeII$ and solids. Then $k_{\tau}^2 \ll k_i^2, k_{\pm}^2$ and due to Eqs. (7) and (14) we have $k_{\pm z} \approx k_{\pm}$. In this approximation we obtain $f_z \approx f \equiv f_z(\chi, \Theta_{sol} = 0)$ and $\tilde{\delta} \approx \delta \equiv (k_+ - k_-)/ik_g$. Then the dependance of D_i^+ on the incidence angle, from (33), is factorized out and is reduced to the multiplier $\sim \cos \theta_{sol}$:

$$D_i^+(\chi, \theta_{sol}) \approx \frac{4Z_g \cos \theta_{sol}}{f^2 + \delta^2} \times \left(\frac{k_{iz}}{k_g} \frac{(k_{iz} + k_{jz})(k_{iz} + k_{kz})}{(k_{iz} - k_{jz})(k_{iz} - k_{kz})} \right) \Big|_{\theta_{sol}=0}. \quad (37)$$

The frequency dependence of the transmission factors, at normal incidence $\theta_{sol}=0$, is shown in Fig.5. The relative creation probabilities of phonons, R^- , and R^+ rotons are determined by the multipliers in parenthesis in (37). They can be rewritten in terms of k_i , while taking care of the signs: k_{1z} and k_{3z} at $\theta_{sol}=0$ are equal to k_1 and k_3 , but k_{2z} at $\theta_{sol}=0$ is equal to $(-k_2)$ (because of the negative group velocity of R^- rotons). Then for $i=2$ we obtain

$$D_2^+ \propto \frac{k_2 - k_1}{k_2 + k_1} \cdot \frac{k_3 - k_2}{k_3 + k_2} \cdot k_2. \quad (38)$$

Both the first and second multipliers here are less than unity. In the analogous expressions for $D_{1,3}^+$ one of the

two corresponding multipliers is reversed. So, for the ratio $D_2^+/D_{1,3}^+$, the effect is squared and we obtain

$$D_2^+ \ll D_{1,3}^+. \quad (39)$$

Near the roton minimum, when $\chi \rightarrow \chi_{rot}$, the R^- and R^+ roton branches merge, their group velocities tend to zero, and so do their creation probabilities $D_{2,3} \rightarrow 0$ (because they are proportional to the energy density fluxes, which are proportional to the group velocities). In Eq. (38) the multiplier $(k_3 - k_2)$ comes from the group velocity. In the same way $D_{1,2}^+ \rightarrow 0$ near the maxon maximum $\chi \rightarrow \chi_{max}$, where the phonon and R^- roton branches merge. Thus D_2^+ becomes zero at both ends of the frequency interval in which D_2^+ is defined. Both the strong inequality (39) and asymptotic behavior of D_2 at $\chi \rightarrow \chi_{rot}, \chi_{max}$ (see Fig.5) are the consequences of the simple relations $0 < k_{1z} < (-k_{2z}) < k_{3z}$ from Eqs. (20), which reflect the qualitative behavior of the dispersion curve for superfluid helium, as shown on Fig.1.

The creation probability of R^- rotons at the interface is very small for all energies. It should also be noted that, at low temperatures, the main contribution to the energy flow through the interface is due to phonons of energies less than the roton gap (i.e. with $\chi < \chi_{rot}$), which are not yet taken into account. Thus we have a convincing explanation why R^- rotons were not detected in experiments which created beams of quasiparticles in helium, by a solid heater, as for example [2].

The expressions for D_i^+ (33), (37), and D_{\rightarrow} (34) are written as functions of the incidence angle or k_{τ} . What can be measured experimentally are the energy flows as functions of transmission angles. If phonons be incident on the interface isotropically, then as the transmission angle for each wave is defined by Eqs. (23), the quasiparticles of each type are transmitted in a narrow cone with the cone angle twice the θ_i^{cr} . Thus the phonons are injected into the helium in the widest cone and R^+ rotons in the narrowest cone. For the total transmission coefficient as function of transmission angle θ we obtain

$$D_{\rightarrow}(\chi, \Theta) = \sum_{i=1}^3 D_i^+(\chi, k_{\tau i} = k_i \sin \theta), \quad (40)$$

where $D_i^+(\theta) = 0$ for $\theta > \theta_i^{cr}$

and $k_{\tau i}$ are the transverse components of the wave vectors of wave i transmitted at angle θ .

B. Phonon or roton in the helium incident on the interface

Now let us consider one of the quasiparticles of superfluid helium, phonon or roton, incident on the interface. In terms of plane waves, a wave i of frequency ω and wave vector of length $k_i(\omega)$ is incident. The solution in the solid consists of only one transmitted wave, and solution in the quantum fluid consists of one incident wave i and three reflected waves $j=1, 2, 3$; it can be represented as a sum of solutions P_{out} and $P_{in}^{(i)}$ from Eqs. (21) and (22) (see Fig.2). The boundary conditions (8) enable us to express all the amplitudes through the amplitude of the incident wave, and thus to obtain the nine amplitude reflection coefficients r_{ij} . The coefficient r_{ij} is the ratio of the pressure amplitudes of the reflected wave j to the

incident wave i for $i, j = 1, 2, 3$:

$$r_{ii} = -\frac{\psi_i}{\psi_i^*} \cdot \frac{f_{-2z}^{(i)}}{f_{2z}} \cdot \frac{f_{-z}^{(i)} + Z - i\tilde{\delta}}{f_z + Z - i\tilde{\delta}}; \quad (41)$$

$$r_{ij} = 2 \frac{\psi_j}{\psi_i^*} \cdot \frac{k_{iz}(k_i^2 - k_k^2)}{f_{2z}} \cdot \frac{k_{kz}/k_g + Z - i\tilde{\delta}}{f_z + Z - i\tilde{\delta}} \cdot \varepsilon_{ijk}. \quad (42)$$

Here the subscripts take values $\{i, j, k\} = \{1, 2, 3\} + \text{perm.}$; ε_{ijk} is the Levi-Civita symbol, equal to 1 if $\{i, j, k\} = \{1, 2, 3\}, \{2, 3, 1\}$, or $\{3, 1, 2\}$ and to (-1) if $\{i, j, k\} = \{2, 1, 3\}, \{1, 3, 2\}$, or $\{3, 2, 1\}$; $Z = -Z_g k_{sol} z / k_{sol}$ is the generalization of definition (28). For the incidence angles less than critical $k_\tau^2 < k_{sol}^2$, $k_{sol} z < 0$ and Z is given by Eq. (28). For greater incidence angles the new notation must be used, as $\cos \theta$ is not defined. Then $k_{sol} z \in \mathbf{C}_-$ for the wave to be damped in $z < 0$ and therefore $Z = i|Z|$. The constructions $f_{-nz}^{(i)}$ are

$$f_{-z}^{(i)} = f_{-3z}^{(i)} / (k_g f_{-2z}^{(i)}), \quad (43)$$

$$f_{-nz}^{(i)} = f_{nz}[k_{1z}, k_{2z}, k_{3z}]|_{k_{iz} \rightarrow (-k_{iz})}$$

for $i = 1, 2, 3$; $n = 2, 3$.

The amplitude coefficient of transmission t_i^\leftarrow for the incident wave of type i is

$$t_i^\leftarrow = \frac{(k_{iz} + k_{jz})(k_{iz} + k_{kz})}{\psi_i^*} \cdot \frac{2k_{iz}/k_g}{f_z + Z - i\tilde{\delta}}. \quad (44)$$

Then the energy transmission coefficient D_i^\leftarrow for the wave i can be calculated as the fraction of the energy of the incident wave packet that is transmitted into the solid. It is explicitly shown that

$$D_i^\leftarrow(\chi, k_\tau) = D_i^\rightarrow(\chi, k_\tau). \quad (45)$$

This important relation ensures thermodynamic equilibrium between the solid and helium at equal temperatures on both sides of the interface. Due to Eq. (45) we can from now on omit the arrows in the sub- and superscripts of D_i and D .

The reflection coefficients for $i = j$ are just $R_{ii} = |r_{ii}|^2$ and from Eq. (41) we obtain

$$R_{ii} = \left| \frac{(Z - i\tilde{\delta})k_g f_{-2z}^{(i)} + f_{-3z}^{(i)}}{(Z - i\tilde{\delta})k_g f_{2z} + f_{3z}} \right|^2. \quad (46)$$

For $i \neq j$ we have to take into account that energy flows for all waves are proportional to group velocities (32), and then from Eqs. (42) and (43) we derive

$$R_{ij} = R_{ji} = 4k_g^2 |k_{iz}k_{jz}(k_i^2 - k_k^2)(k_j^2 - k_k^2)| \times \left| \frac{Z - i\tilde{\delta} + k_{kz}/k_g}{(Z - i\tilde{\delta})k_g f_{2z} + f_{3z}} \right|^2. \quad (47)$$

The quantity R_{ij} is the probability of quasiparticle j being created at the interface when quasiparticle i is incident, so the R_{ij} can be also called "conversion coefficients".

Their dependence on frequency at normal incidence is shown on Fig. 6. We see, in particular, that at the roton minimum $\chi \rightarrow \chi_{rot}$, where the R^- and R^+ roton branches merge, these quasiparticles are reflected into each other with probability that tends to unity $R_{23} \rightarrow 1$. The same

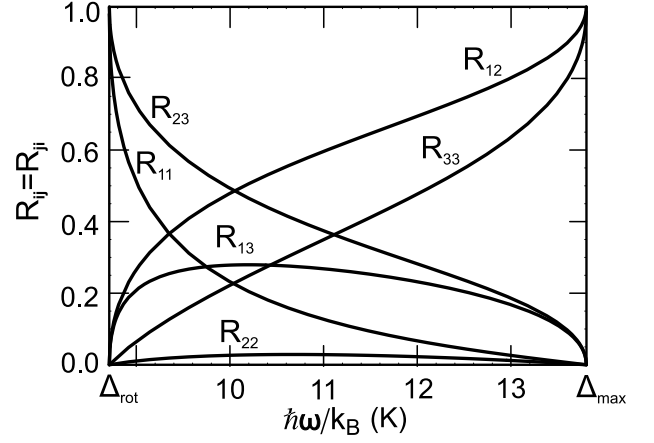


FIG. 6. Functions $R_{ij}(\omega, \theta_i = 0)$ for $i, j = 1, 2, 3$.

effect is present for phonons and R^- rotons at the maxon maximum.

The angular dependence of R_{ij} is most easily analyzed in terms of k_τ instead of the three angles of incidence. The values of k_τ equal to $k_{sol}(\omega)$ or $k_i(\omega)$ correspond to different critical angles of incidence. So, when $k_\tau \in (0, k_{sol})$, the quantities Z and $f_{\pm nz}^{(i)}$ (i.e. f_{nz} and f_{-nz} for $i = 1, 2, 3$) are all real, so all the waves are traveling waves and $D \neq 0$. When $k_\tau \in (k_{sol}, k_1)$, the wave in the solid is damped, $Z = i|Z|$ and $D = 0$, but $f_{\pm nz}^{(i)} \in \mathbf{R}$ and all the waves in the helium are still reflected into each other. When $k_\tau \in (k_1, k_2)$, the phonon wave in helium is damped $k_{1z} = i|k_{1z}|$ and no longer gives a traveling wave packet, and $f_{\pm nz}^{(i)}$ also become complex. This corresponds to R^\pm rotons incident at angles greater than $\theta_{31,21}^{cr}$ and reflecting into themselves or into each other. When $k_\tau \in (k_2, k_3)$ the quantities $f_{\pm nz}^{(i)}$ are also complex but the structure is different; this case corresponds to R^+ rotons incident at angles greater than θ_{32}^{cr} and reflecting into R^+ rotons, again with probability 1.

In all the cases, energy conservation can be explicitly verified but it takes different forms:

$$\begin{aligned} k_\tau < k_{sol}(\omega) : \sum_{j=1}^3 R_{ij} &= 1 - D_i \quad \text{for } i = 1, 2, 3; \\ k_{sol}(\omega) < k_\tau < k_1(\omega) : \sum_{j=1}^3 R_{ij} &= 1 \quad \text{for } i = 1, 2, 3; \\ k_1(\omega) < k_\tau < k_2(\omega) : R_{22} &= R_{33} = 1 - R_{23}; \\ k_2(\omega) < k_\tau < k_3(\omega) : R_{33} &= 1. \end{aligned} \quad (48)$$

For the interface between helium and a solid, the limit $Z_0 \ll 1$ is a good approximation, and in the Eqs. (41), (42) and (46), (47) Z can be neglected (in this limiting case $D \rightarrow 0$ and $\Theta_i^{cr} \rightarrow 0$). However, the angles are not small anymore, as was the case for D_i , and the angular dependence of the coefficients is strong. This can be clearly seen in Fig. 7, where the graphs of R_{1j} and R_{2j} are shown for $\hbar\omega/k_B = 10K$. The coefficient R_{21} becomes zero at the critical angle θ_{21}^{cr} . The peak of R_{22} and minimum of R_{23} correspond to angles above critical, where k_{1z} is imaginary and the damping depth of the phonon wave becomes roughly half of the damping depth of the nonlocality kernel $h(r)$; then the imaginary part of the

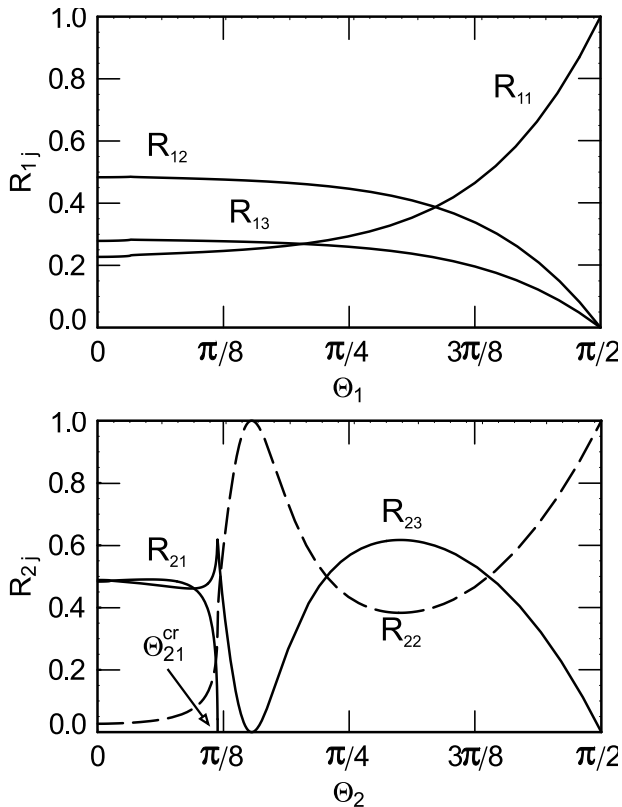


FIG. 7. [(a) and (b)] Reflection coefficients R_{ij} and R_{2j} for $j = 1, 2, 3$ as functions of incidence angle θ at $\chi = 0.2871$ ($\hbar\omega/k_B = 10K$).

numerator of Eq. (47) turns to zero, and as Z is small, R_{23} has a deep minimum.

A more extensive analysis of the functions $R_{ij}(\chi, \theta_i)$ for the case $Z_0 \ll 1$ allows us to state the following: the main processes near the roton minimum $\chi \rightarrow \chi_{rot} + 0$, for all angles, are the conversion of R^- and R^+ rotons into each other and reflection of phonons into themselves; near the maxon maximum phonons and R^- rotons are converted into each other and reflection of R^+ rotons into themselves. For phonons and R^- rotons, when the incidence angle becomes close to $\pi/2$, the probabilities of reflection into themselves $R_{11,22}$ tend to unity; for R^+ rotons this happens at $\theta_3 \rightarrow \theta_{32}^{cr} - 0$, and at greater angles $R_{33} = 1$ exactly. The conversion coefficients $R_{1j,j1}$ for $j = 2, 3$ are monotonically decreasing functions of the angles of incidence; R_{1j} becomes zero at $\theta_j \rightarrow \pi/2$ as $\sqrt{\pi/2 - \theta_1}$, R_{j1} at $\theta_j \rightarrow \theta_{j1}^{cr}$ as $\sqrt{\theta_{j1}^{cr} - \theta_j}$. A little above θ_{j1}^{cr} the coefficients R_{22} and R_{33} have high sharp peaks, and R_{23} a corresponding minimum, as described above.

Then for the case depicted in Fig. 4, the most powerful beam will be always beam *ii* (basically because of phonons with energies less than χ_{rot}). We have shown that R^- rotons are hardly created by a solid heater (39), and the probability of R^+ rotons creation is also quite small at frequencies near χ_{rot} (see Fig. 5) if the incident beam consists mainly of low energy phonons. It was shown in [7, 8] that in a phonon beam, low energy phonons (l-phonons) are converted into phonons with energy about 10K (h-phonons). The fraction of the energy in the initial beam, which is converted to the h-phonons, can be up to 50% [8]. The conversion coefficient of these phonons to R^- rotons is given by R_{12} at $\hbar\omega/k_B \approx 10K$, it is much greater than at the roton minimum and almost

reaches 1/2 at normal incidence, which is more than R_{11} , see Fig. 7.

We suggest the experimental setup depicted in Fig. 8. The heater injects a phonon beam, in which h-phonons are created. The h-phonons, incident on the solid-helium interface, are reflected into three beams of phonons, R^- rotons and R^+ rotons of comparable intensities (the R^- rotons are reflected backwards). These beams propagate towards the free surface of helium, and quantum evaporate atoms from it (the R^- rotons evaporate atoms backward [3]), which are then detected. Thus the energy is transported from the heater to the interface by phonons, and then to the detector by R^- rotons along a Z-shaped trajectory, with retro-reflection at the point of creation of R^- rotons and retro-refraction on the surface. The angles and fractions of the initial beam's energy, which is transferred to different reflected beams, are shown for the h-phonon part of the incident beam. The l-phonons for the most part are directly reflected and are not shown.

If the source of quasiparticles has more rotons in the incident beam, as the one used in [3], then beams 32 and 23 may become also be detectable.

The main contribution to the energy flow through the interface at low temperatures can be expected to be made by phonons below the roton gap, i.e. with $\chi \in (0, \chi_{rot})$. The problem of transmission through interfaces by phonons with anomalous dispersion was solved in Refs. [30] and [20]. In the current work, the dispersion relation (1) that is used is more general than the one used in the previous works, it is non-monotonic and normal below the roton gap.

When $\chi < \chi_{rot}$, the roots $k_{2z,3z}$ are defined so that $k_{1z} > 0$ and $k_{3z} = -k_{2z}^* \in \mathbf{C}_+$, as shown in section 2.3. With these k_{iz} the out- and in-solutions are constructed (21), (22). So the amplitude coefficients are still defined by Eqs. (27) and (31), but the quantities $f_{\pm nz}^{(i)}$ are now complex. The only valid reflection coefficient R_{11} is defined by (46) with the complex k_{iz} as introduced above. The transmission coefficient is $D_{ph} = 1 - R_{11}$. It can be shown that in the limit of small frequencies $\chi \rightarrow 0$, when the dispersion is almost linear, the expression for D_{ph} approaches the standard one for linear dispersion. At $\chi \rightarrow \chi_{rot} - 0$ it decreases rapidly to less than half because of the increasing influence of the roton waves. The curve $D(\chi)$ is continuous at χ_{rot} but has a kink.

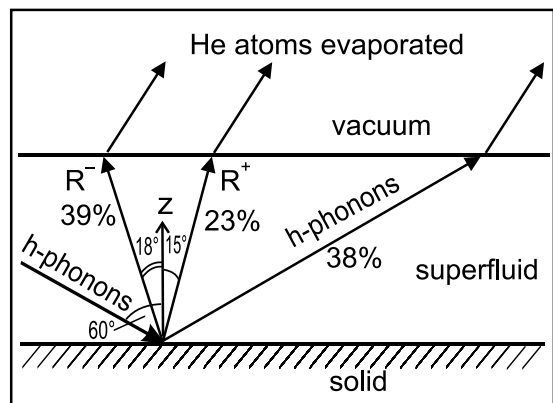


FIG. 8. The predicted creation of R^- rotons by h-phonons incident on the interface with a solid. There should be backward reflection and quantum evaporation with backward refraction.

IV. ENERGY FLOWS THROUGH THE INTERFACE

When a phonon in the solid, of frequency ω and wave vector \mathbf{k}_{sol} , is incident on the interface at angle θ_{sol} , the average energy transferred into helium is $\hbar\omega D(\omega, k_\tau)$, where D is given by (34) and $k_\tau = k_{sol} \cos \theta_{sol}$. Let the phonons in the solid be in thermodynamic equilibrium at temperature T . Then the normal component of the density of energy flow through the interface is (see for example [11])

$$Q(T) = \int \frac{d^3 k_{sol}}{(2\pi)^3} \hbar\omega n_T(\omega) s_{sol} \cos \theta_{sol} D, \quad (49)$$

where n_T is the Bose-Einstein distribution function and the integration domain is the half-space $k_{sol,z} > 0$. The parts of this energy flow, that are transferred into helium by either phonons, or R^- , or R^+ rotons of helium that are created at the interface by the incident phonons, are obtained in the same way. But instead of the full coefficient D , we now use the partial transmission coefficients D_i . These are the corresponding creation probabilities of the quasiparticles. After changing the integration variables to the arguments of $D_i(\omega, k_\tau)$, the partial energy flows can be expressed in the form

$$Q_i^{\rightarrow}(T) = \int \frac{d\omega}{8\pi^2} \hbar\omega n_T(\omega) \int_0^{k_i^2(\omega)} dk_\tau^2 D_i(\omega, k_\tau). \quad (50)$$

Here the upper limit by k_τ^2 corresponds to the maximum transmission angle of quasiparticles of type i , equal to θ_i^{cr} from (25). The quantities Q_i^{\rightarrow} for $i=1, 2, 3$, are the individual contributions of phonons, R^- , and R^+ rotons, to the energy flux from the solid into helium.

Their contributions to the energy flux, in the opposite direction $Q_i^{\leftarrow}(T)$, are the normal components of the energy fluxes from helium into the solid. These are realized by helium quasiparticles of type i incident on the interface. The average energy transferred into the solid per incident quasiparticle of type i is, due to (45), $\hbar\omega D_i(\omega, k_\tau)$. Then the energy flux is derived in the same way as Eq. (49), with the difference that instead of s_{sol} in the integral we have $|u_i|$, because the number of quasiparticles incident on the interface per unit of time is proportional to their group velocity:

$$Q_i^{\leftarrow}(T) = \int \frac{d^3 k_i}{(2\pi)^3} \hbar\omega n_T(\omega) |u_i| \cos \Theta_i D_i. \quad (51)$$

When changing the integration variables to (ω, k_τ) , we use the Jacobian determinant and we obtain explicitly that $Q_i^{\leftarrow}(T) = Q_i^{\rightarrow}(T)$.

Fig.9a shows the ratio between the contributions to the energy flux through the interface of all quasiparticles above the roton gap (i.e. with $\hbar\omega/k_B > \Delta$) $Q_> = Q_1 + Q_2 + Q_3$, and the contribution of phonons below the roton gap $Q_<$, which is obtained using (50) with D_{ph} . We see that at temperatures $T < 1$ K the phonons are dominant. However, at $T \approx 2.5$ K the two contributions are equal, and at higher temperatures the phonons and rotons above the roton gap play the main role in heat exchange with the solid, see Fig.9b. The contribution of the R^+ rotons to $Q_>$ increases with temperature and at

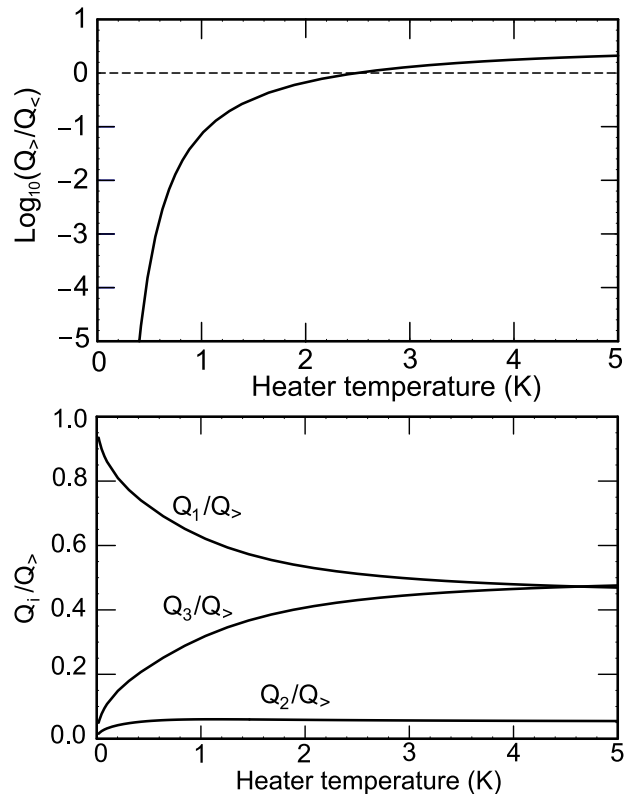


FIG. 9. (a) The ratio of the contributions to the energy flow through the interface by quasiparticles above roton minimum, $Q_>$, to the phonons below roton minimum, $Q_<$, as a function of temperature (on a logarithmic scale). (b) The contributions of phonons, R^- rotons and R^+ rotons to the energy flow created by quasiparticles above the roton minimum, as functions of T

$T \approx 3$ K their contribution surpasses that of the phonons, see Fig.9b. The contribution of the R^- rotons, is approximately constant and is no greater than 6%. This is due to the low creation probability D_2 for all frequencies (39). At $T = 3$ K the contribution of the R^- rotons, to the full energy flow, is 3%.

When there is energy flow through the interface, it induces the Kapitza temperature jump at the interface (see for example [11]). The contributions of quasiparticles of each type to this jump are obtained by differentiating Eq. (50) with respect to T .

V. CONCLUSION

In this work we have solved the problem of the interaction of $HeII$ quasiparticles, i.e. phonons, R^- rotons, and R^+ rotons, with the interface between helium and a solid. These excitations have the non-monotonic dispersion curve shown in (Fig.1). The consistent solution of the problem has been introduced, which allows us to rigorously describe the simultaneous creation of the three types of $HeII$ quasiparticles by any one of them, or by a phonon in the solid which is incident on the interface.

When a phonon in the solid is incident on the interface, it is reflected with some probability and a phonon, R^- roton, or R^+ roton are created with the corresponding probabilities, in the helium. It is shown that the created R^- roton, due to its negative group velocity, is refracted

backward (Fig.3). When some quasiparticle of helium is incident, all the quasiparticles, with the same frequency and transverse wave vectors, are created. The set of six critical angles as functions of frequency are introduced (25), (26). These separate the intervals of angles of incidence for the different quasiparticles, from which other quasiparticles can be created. It is shown when a phonon or R^+ roton is incident, the R^- roton is retro-reflected (i.e. reflected backwards), and likewise when an R^- roton is incident, the phonon and R^+ rotons are retro-reflected (Fig.4). This effect is the Andreev reflection of phonons and rotons.

The probabilities of creation of all quasiparticles at the interface, when any quasiparticle is incident, are derived as functions of frequency and incidence angles (33), (46), (47) and Figs.5,6,7. It is shown that the creation probability of an R^- roton by a phonon in the solid, and vice versa, is very small for all angles and frequencies (38), (39). This means that R^- rotons are as badly created

by a solid heater as they are poorly detected by a solid bolometer. This explains the failure to detect R^- rotons in direct experiments until 1999 [3]. New predictions are made for experiments with beams of phonons and rotons interacting with the solid interface, and in particular, creating R^- rotons at the interface by a beam of high-energy phonons (h-phonons).

The full energy flow through the interface, is also calculated as a function of temperature of the solid, as well as the individual contributions of the phonons, R^+ , and R^- rotons (50) to it, see Fig.9. The contribution of the R^- rotons is shown to be very small.

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